Tutorial 4 (Feb 25, 27)

Leon Li

ylli@math.cuhk.edu.hk

Q1) Find the mass M and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the cube

$D = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le x \le a;\ 0 \le y \le a;\ 0 \le z \le a\}$ with density function $\delta(x,y,z) = x^2+y^2+z^2$.

Sol) Idea: Compute relevant triple integrals involved in the definition.

① Mass: $M = \iiint_D \delta(x,y,z)\, dV = \int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)\, dz\, dy\, dx$

Method 1: Direct computations.

$$= \int_0^a \int_0^a \left[(x^2+y^2)z + \frac{z^3}{3}\right]_0^a dy\, dx = \int_0^a \int_0^a \left(ax^2 + ay^2 + \frac{a^3}{3}\right) dy\, dx$$

$$= a\int_0^a \left[\left(x^2 + \frac{a^2}{3}\right)y + \frac{y^3}{3}\right]_0^a dx = a\int_0^a \left(ax^2 + \frac{a^3}{3} + \frac{a^3}{3}\right) dx$$

$$= a^2\left[\frac{x^3}{3} + \frac{2}{3}a^2 x\right]_0^a = a^2\left(\frac{a^3}{3} + \frac{2}{3}\cdot a^3\right) = a^5.$$

Method 2: Split the triple integral into a sum of triple integrals.

$$= \int_0^a\int_0^a\int_0^a x^2\, dz\, dy\, dx + \int_0^a\int_0^a\int_0^a y^2\, dz\, dy\, dx + \int_0^a\int_0^a\int_0^a z^2\, dz\, dy\, dx = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

Note that $\mathrm{I} = \left(\int_0^a dz\right)\left(\int_0^a dy\right)\left(\int_0^a x^2\, dx\right) = a \cdot a \cdot \frac{a^3}{3} = \frac{a^5}{3}$

Similarly, $\mathrm{II} = \left(\int_0^a dz\right)\left(\int_0^a y^2\, dy\right)\left(\int_0^a dx\right) = a \cdot \frac{a^3}{3} \cdot a = \frac{a^5}{3}$

$\mathrm{III} = \left(\int_0^a z^2\, dz\right)\left(\int_0^a dy\right)\left(\int_0^a dx\right) = \frac{a^3}{3} \cdot a \cdot a = \frac{a^5}{3}$

$\therefore\ M = \mathrm{I} + \mathrm{II} + \mathrm{III} = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3} = a^5$

② Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$, where

ⓘ $M_{yz} = \int_0^a \int_0^a \int_0^a x(x^2+y^2+z^2)\,dz\,dy\,dx$

ⓘⓘ $M_{xz} = \int_0^a \int_0^a \int_0^a y(x^2+y^2+z^2)\,dz\,dy\,dx$

ⓘⓘⓘ $M_{xy} = \int_0^a \int_0^a \int_0^a z(x^2+y^2+z^2)\,dz\,dy\,dx$

Adopting the idea of method 2:

ⓘ $= \int_0^a \int_0^a \int_0^a x^3\,dz\,dy\,dx + \int_0^a \int_0^a \int_0^a xy^2\,dz\,dy\,dx + \int_0^a \int_0^a \int_0^a xz^2\,dz\,dy\,dx$

$= a \cdot a \cdot \frac{a^4}{4} + a \cdot \frac{a^3}{3} \cdot \frac{a^2}{2} + \frac{a^3}{3} \cdot a \cdot \frac{a^2}{2} = a^6\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{6}\right) = \frac{7a^6}{12}$

ⓘⓘ $= \int_0^a \int_0^a \int_0^a y^3\,dz\,dy\,dx + \int_0^a \int_0^a \int_0^a yx^2\,dz\,dy\,dx + \int_0^a \int_0^a \int_0^a yz^2\,dz\,dy\,dx$

$= a^6\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{6}\right) = \frac{7a^6}{12}$

ⓘⓘⓘ $= \frac{7a^6}{12}$ (Similar argument as ⓘ, ⓘⓘ)

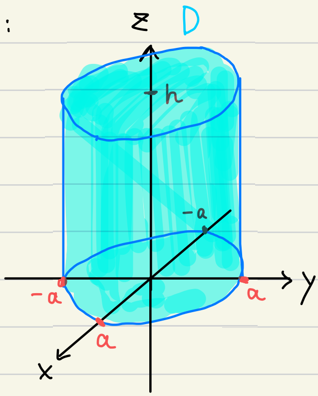
$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\frac{7a^6}{12}}{a^5}, \frac{\frac{7a^6}{12}}{a^5}, \frac{\frac{7a^6}{12}}{a^5}\right) = \left(\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12}\right)$

Q2) Find the moment of inertia of the solid cylinder D with radius a and height h with constant density δ about its axis.

Sol) Idea: Formulate the problem suitably and compute using cylindrical coordinates.

Step 1: Put D into $\mathbb{R}^3$ in a suitable way.

Picture:



Step 2: Describe D in cylindrical coordinates.

$$D = \{(r,\theta,z) \in [0,+\infty) \times [0,2\pi) \times \mathbb{R} \mid 0 \leq r \leq a,\ 0 \leq \theta < 2\pi,\ 0 \leq z \leq h\}$$

Step 3: Compute the moment of inertia about the z-axis.

$$I_z = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV = \int_0^h \int_0^{2\pi} \int_0^a r^2\,\delta\,(r\,dr\,d\theta\,dz)$$

$$= \delta \cdot \left(\int_0^h dz\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^a r^3\,dr\right)$$

$$= \delta \cdot h \cdot 2\pi \cdot \frac{a^4}{4} = \frac{\pi}{2}\delta h a^4$$

Q3) Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}}\,dz\,dy\,dx.$

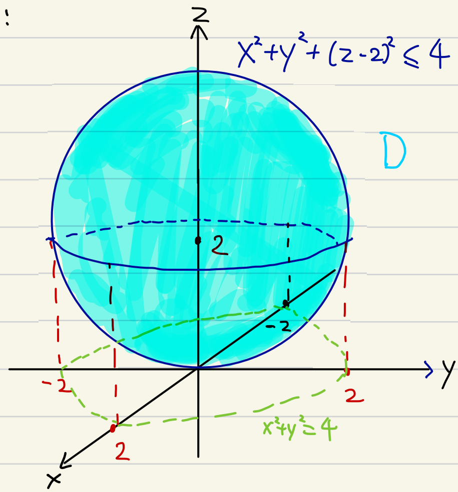
Sol) Idea: Understand the integral better and apply a change of coordinates.

Step 1: Describe the domain of integration D in $\mathbb{R}^3$.

$D=\{(x,y,z)\in\mathbb{R}^3 \mid -2\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\ 2-\sqrt{4-x^2-y^2}\le z\le 2+\sqrt{4-x^2-y^2}\}$

Step 2: Sketch D in $\mathbb{R}^3$.

Picture:



Step 3: Describe D in terms of spherical coordinates

$$\begin{cases} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{cases}$$

Note that $x^2+y^2+(z-2)^2\le 4 \iff x^2+y^2+z^2\le 4z$

In spherical coordinates: $\rho^2\le 4\rho\cos\phi \iff \rho\le 4\cos\phi$

$\therefore\ D=\{(\rho,\phi,\theta)\in[0,+\infty)\times[0,\pi]\times[0,2\pi) \mid 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2},\ 0\le\rho\le 4\cos\phi\}$

Step 4: Evaluate the integral using spherical coordinates.

$$\text{Integral} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{4\cos\phi} (\rho^2)^{\frac{3}{2}} \cdot (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{2}} \int_0^{4\cos\phi} \rho^5 \sin\phi\, d\rho\, d\phi \right)$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{2}} \left[\frac{\rho^6}{6} \cdot \sin\phi \right]_0^{4\cos\phi} d\phi$$

$$= 2\pi \cdot \frac{4096}{6} \int_0^{\frac{\pi}{2}} \cos^6\phi \sin\phi\, d\phi$$

$$= \frac{4096}{3}\pi \left[-\frac{\cos^7\phi}{7} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{4096}{3}\pi \left(\frac{1}{7} \right) = \frac{4096}{21}\pi$$